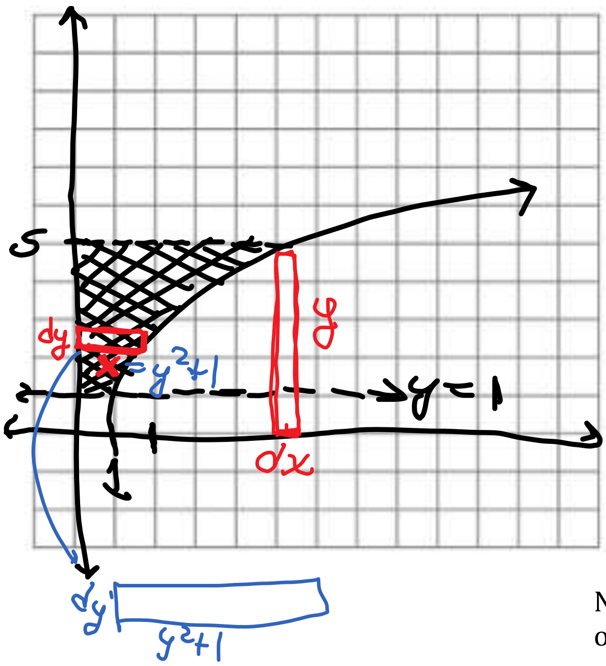
An alternate way to find areas under curves:

Example: Find the area of the region bounded by the curve $y=\sqrt{x-1}$, the y-axis and the lines y=1 and y=5.

Sketch first:



Note that the bases of the rectangles that will comprise the area are now horizontal (meaning they are x-values) and their infinitely small heights are given by *dy*.

Since *dy* will be your variable of integration, and your function is currently in terms of *x*, it will be wise to re-write the function in terms of y. In other words, solve the curve for *x*.

$$(y)^2=\left(\sqrt{x-1}\right)^2$$
$$y^2=x-1 \longrightarrow x=y^2+1$$

You now have a function which will give you the base of any horizontally-aligned rectangle for a given y-value.

Now, set up an integral with respect to *y* which will sum up the areas of all of these rectangles:

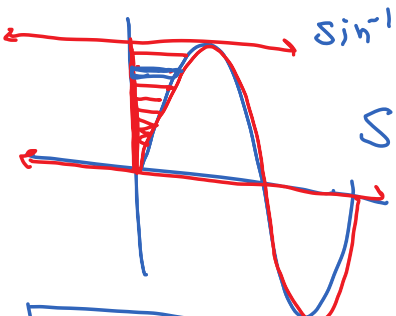
$$\int_1^5 (y^2+1)\,dy$$

$$\int_a^b f(x) = F(x)\Big]_a^b$$
$$F(b)-F(a)$$

Evaluate the integral and you have your answer.

$$\left[\frac{1}{3}y^3+y\right]_1^5 = \frac{1}{3}(5)^3+5-\left(\frac{1}{3}(1)^3+1\right)$$
$$\frac{136}{3}\approx\boxed{45.333}$$

Practice: Find the area of the region bounded by the functions y = sin(x), y=0, and y=1, $x=0$.



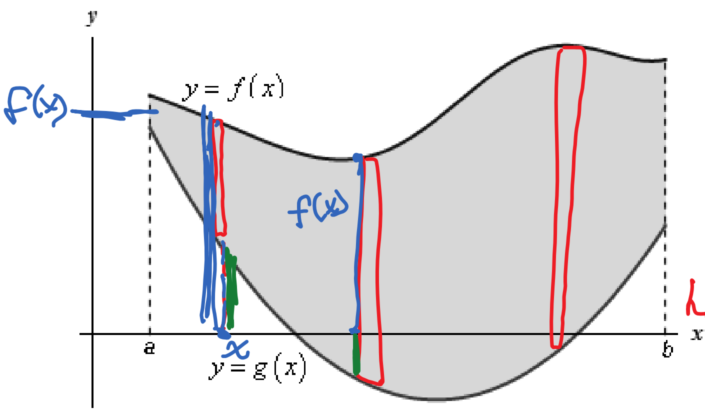
$$\sin^{-1}(y)=\sin(x))\sin^{-1}$$
$$\sin^{-1}(y)=x$$

$$\int_0^1 \sin^{-1}(y)\cdot dy$$
$$\boxed{\approx 0.571}$$

$$\boxed{\frac{d}{dx}\sin^{-1}(x)=\frac{1}{\sqrt{1-x^2}}}$$

Area Between Two Curves

Let us consider the region bounded by two functions on top and bottom (and, as we'll see, left and right) as below:



Here we are not so lucky as to draw a rectangle from the x-axis whose height is merely the y-value. Instead, the height of the rectangle varies depending on where it is.

Sketch several representative rectangles that span the entire height of the shaded region, with base *dx*.

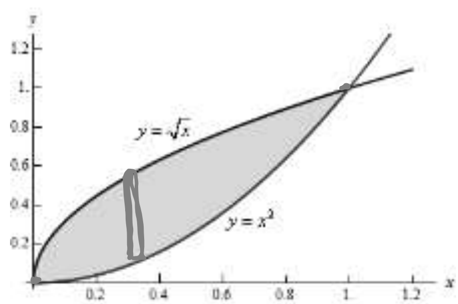
How do you represent the height of these rectangles?

height = $f(x) - g(x)$

That leads us to this general formula:

$\int_a^b \left(\text{upper function} - \text{lower function}\right) dx$ for vertical rectangles or $\int_c^d \left(\text{Right function} - \text{Left function}\right) dy$ for horizontal.

Practice: Find the area of the region bounded by y=x² and y = $\sqrt{x}$.



Note that the bounds of integration are not given. How do you find them?

$(x^2 = \sqrt{x})^2 \quad x^4 = x \rightarrow x^4 - x = 0 \quad x(x^3-1) = 0 \Rightarrow \underline{x=1}$

Which is your upper function? Which is the lower?

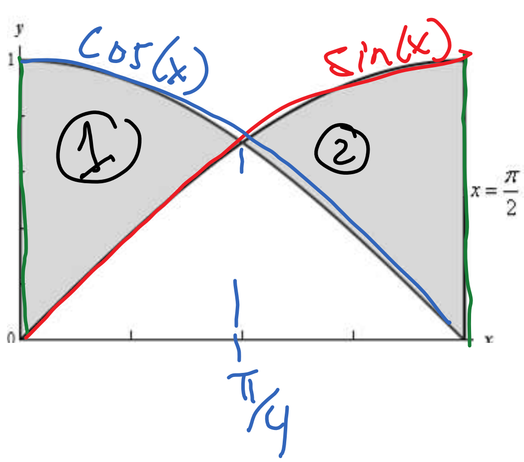
upper = $\sqrt{x}$ lower = x^2

Set up, *simplify*, then integrate:

$$\int_0^1 x^{1/2} - x^2 \, dx = \frac{2}{3}x^{3/2} - \frac{1}{3}x^3 \Big]_0^1 = \frac{2}{3}(1)^{\cancel{3/2}} - \frac{1}{3}(1)^{\cancel{3}} - \cancel{(0 - 0)}$$

$$\frac{1}{3} = \frac{2}{3} - \frac{1}{3}$$

Practice 2: Determine the area of the region bounded by y=sin(x), y=cos(x), and the lines x=0 and x=$\frac{\pi}{2}$.



Be careful! Are the upper and lower functions always the same? Maybe you will need more than one integral to solve this thing.

$$A_1 = \int_0^{\pi/4} \cos(x) - \sin(x) \, dx$$

$$\sin(x) + \cos(x) \Big]_0^{\pi/4}$$

$$\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) - \left(\underbrace{\cancel{\sin 0}}_{0} + \underbrace{\cancel{\cos 0}}_{1}\right)$$

$$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1 = \sqrt{2} - 1$$

$$A_2 = \int_{\pi/4}^{\pi/2} \sin(x) - \cos(x)\,dx$$

$$-\cos(x) - \sin(x) \Big]_{\pi/4}^{\pi/2}$$

$$-\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) - \left(-\cos(\pi/4) - \sin\left(\frac{\pi}{4}\right)\right)$$

$$\cancel{-0} - 1 - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)$$

$$-1 - -\sqrt{2}$$

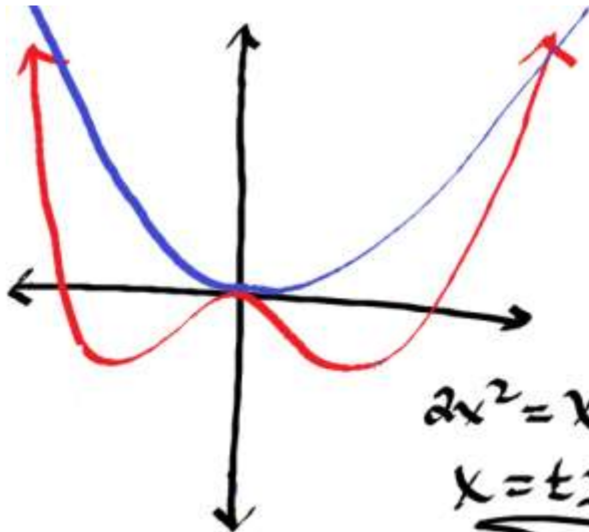
$$-1 + \sqrt{2}$$

$$\sqrt{2} - 1$$

$$A = \overbrace{\sqrt{2}-1}^{1} + \overbrace{\sqrt{2}-1}^{2}$$

$$2\sqrt{2} - 2$$

Worksheet #3:



$$\int_{-2}^{2} \text{upper} - \text{lower}\, dx$$

$$\int_{-2}^{2} 2x^2 - (x^4 - 2x^2)$$

$$\int_{-2}^{2} 2x^2 - x^4 + 2x^2\, dx$$

$2x^2 = x^4 - 2x^2$

$$\int_{-2}^{2} 4x^2 - x^4\, dx$$

$x = \pm 2, 0$

$$\left.\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_{-2}^{2}$$

$$\frac{4}{3}(2)^3 - \frac{1}{5}(2)^5 - \left(\frac{4}{3}(2)^3 - \frac{1}{5}(-2)^5\right)$$

32/3 - 32/5 - (-32/3--32/5)
32/3 - 32/5 + 32/3 - 32/5
64/3 -64/5
320/15 - 192/15
128/15