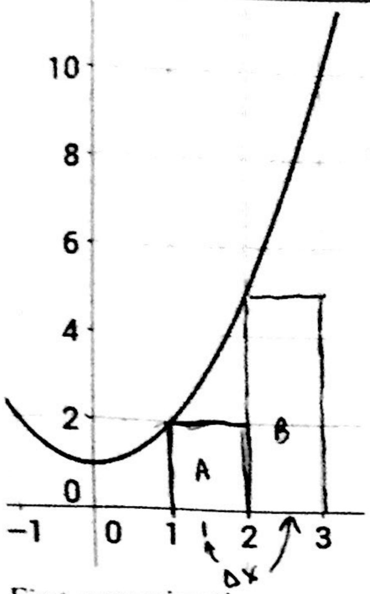
How to find the area under a curve?

Find a method to approximate the area enclosed by $y = x^2 + 1$, the vertical lines x=1 and x=3, and the x-axis.
(This is called 'the area under the curve')



First approximation

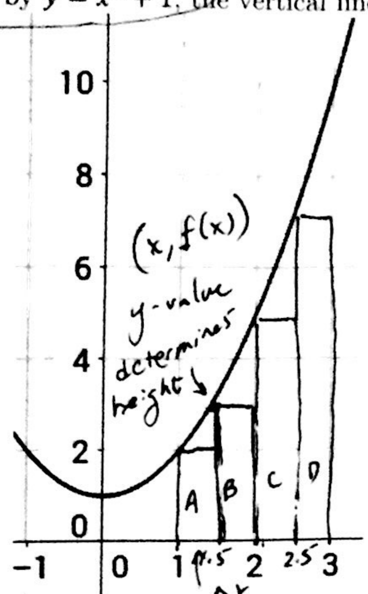
$$\text{Area} \approx \underbrace{1\cdot(2)}_{\text{rect A}} + \underbrace{1\cdot(5)}_{\text{rect B}}$$

base, f(1), or height, base, f(2) or height

$$\approx 2 + 5$$

$$\approx 7 \text{ sq. units}$$

How can you improve your method?
more, smaller rectangles.



Second approximation

$$\text{Area} \approx \underbrace{\tfrac{1}{2}(f(1))}_{\text{Rect. A}} + \underbrace{\tfrac{1}{2}(f(1.5))}_{B} + \underbrace{\tfrac{1}{2}(f(2))}_{C} + \underbrace{\tfrac{1}{2}\cdot(f(2.5))}_{D}$$

Base height

$$\approx \tfrac{1}{2}\cdot 2 + \tfrac{1}{2}(3.25) + \tfrac{1}{2}(5) + \tfrac{1}{2}(7.25)$$

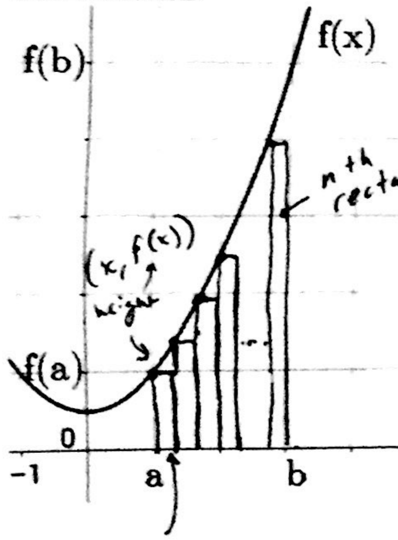
$$\approx \tfrac{1}{2}(2 + 3.25 + 5 + 7.25)$$

$$\approx \tfrac{1}{2}(17.5)$$

$$\approx 8.75 \text{ sq. units}$$

think about $\sum_{i=1}^{4}$

Generalization:



n rectangles; split [a, b]

$$\Delta x = \frac{b-a}{n}$$

Area

1st rectangle: $\Delta x \cdot f(a)$

2nd rectangle: $\Delta x \cdot f(a+\Delta x)$

3rd rectangle: $\Delta x \cdot f(a+2\Delta x)$

⋮

nth rectangle $\Delta x \cdot f(b - \Delta x)$

Sum total

Δx (width, or base)

$$\text{Area} \approx f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_n)\Delta x$$

"a" "b"

$$= \Delta x\,(f(x_0) + f(x_1) + \dots + f(x_n))$$

$$= \Delta x \cdot \sum_{i=1}^{n} f(x_i) \underset{\text{commutative property}}{=} \sum_{i=1}^{n} f(x_i)\cdot \Delta x$$

$$\int_a^b f(x)\cdot dx$$

"S"

$$\sum_{i=1}^{n} f(x_i)\cdot \Delta x$$

Riemann definition of Definite Integral: if f is a continuous function on $[a,b]$ and this interval is equally divided into n intervals of width $\Delta x = \frac{b-a}{n}$, and if $x_i = a + i\Delta x$ is the right endpoint of subinterval i, then:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

← Infinite Rectangles!

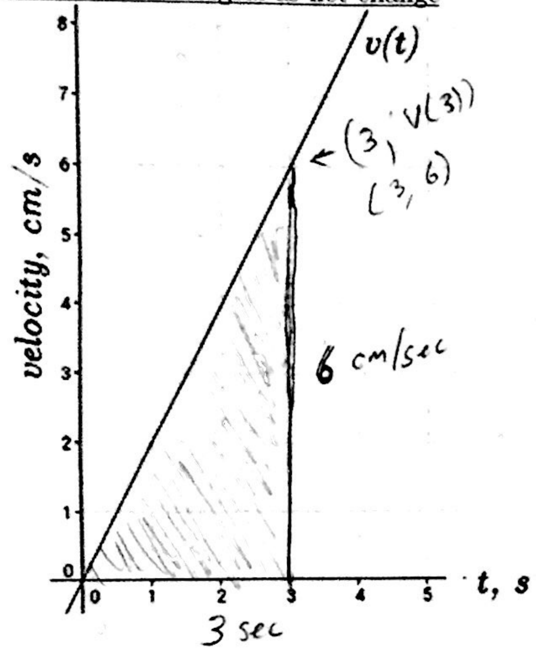
The Fundamental Theorem of Calculus: Part 2

If $f(x)$ is the derivative of $F(x)$, then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

For real numbers a and b (called the limits of integration). It is not required that $a < b$.

The Definite Integral as net change



An object moves along the x-axis such that its velocity in cm/s is given by $v(t) = 2t$. At time $t = 0$ s, the object is at the origin. After 3 seconds, how far as the object traveled?

$x(0) = 0$ $(0, 0)$

1. Find the specific position function $x(t)$.

$x(t) = \int v(t) \cdot dt = \int 2t \cdot dt = t^2 + C$

$x(t) = t^2 + C$

$x(0) = 0 = 0^2 + C$

$0 = C$

$x(t) = t^2 + 0$ cm

2. Use the position function to find the difference between the positions ("displacement") at time $t = 3$ and time $t = 0$.

$x(0) = 0^2 + 0 = 0$ cm

$x(3) = 3^2 + 0 = 9$ cm

Displacement: $x(3) - x(0) \Rightarrow 9\text{cm} - 0\text{cm} = 9\text{cm}$

3. Find the exact area (using geometry) under the velocity function in the same time interval as problem 2. Use units in your calculations.

$A = \frac{1}{2} \cdot B \cdot H$

$\text{Area} = \frac{1}{2}(3\text{ sec})(6\text{ cm/sec})$

$= \frac{1}{2} \cdot 18\ \cancel{\text{sec}} \cdot \frac{\text{cm}}{\cancel{\text{sec}}}$

$= 9$ cm

whoa!

4. Write a *definite* integral that will find the displacement. Then use the second FTC to evaluate the integral.

$\int_0^3 2t\, dt = [t^2]_0^3$

$= 3^2 - 0^2$

$= 9 - 0$

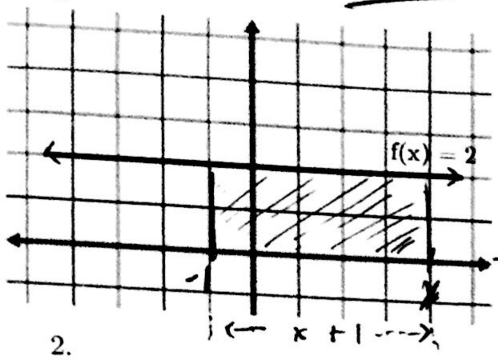
$= 9$

5. In a complete sentence, write a conjecture about what you think the definite integral can be used to find. The definite integral of a rate of change can find the net change of a function.

Connection between Area and Antiderivatives and Slope

For each function, use geometry to find the area A(x) *under the function* f(x) *between* -1 and some arbitrary point x (or, over the interval [-1,x]). *Then, find* A'(x). What do you notice about f(x) and A'(x)?

1.



$f(x) = 2$

Area function $A(x) = \underbrace{(x+1)}_{\text{base}} \cdot \underbrace{2}_{\text{height}}$

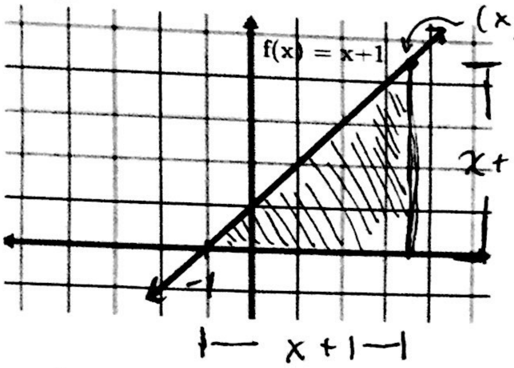
$A(x) = 2x + 2$

$\Downarrow$

A'(x) or $\frac{dA}{dx} = 2$

where we started! So if derivative brings us backwards, then finding the Area is like taking the anti-derivative!!!

2.



$f(x) = x+1$ $\quad \frac{1}{2} \cdot b \cdot h$

Area function $A(x) =$

$= \frac{1}{2}(x+1)(x+1)$

$= \frac{1}{2}(x^2+2x+1)$

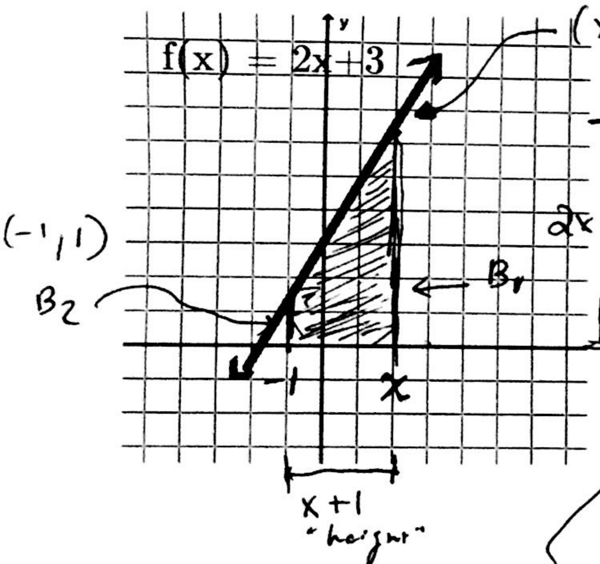
$A(x) = \frac{1}{2}x^2 + x + \frac{1}{2}$

$\Downarrow$ deriv.

A'(x) or $\frac{dA}{dx} = \frac{1}{\cancel{2}}\cancel{2}x + 1$

$= x+1$

3.



$f(x) = 2x+3$

Area function $A(x) =$

$A_{Trapezoid} = \frac{1}{2}(b_1+b_2) \cdot h$

$\frac{1}{2}(\overbrace{2x+3}^{B1} + \overbrace{1}^{B_2}) \cdot (x+1)$

$\frac{1}{2}(2x+4)(x+1)$

$(x+2)(x+1)$

$A(x) = x^2 + 3x + 2$

A'(x) or $\frac{dA}{dx} =$

$f(x) = A'(x) = 2x+3$ ← where we started!

$$\boxed{\int f(x)\,dx = A(x)}$$
Antiderivative = Area

Now go back and find the area under the curve using the FTC:

1. $\int_{-1}^{x} 2\,dx = [2x]_{-1}^{x} = 2x - (2)(-1) = 2x - -2 = 2x+2$ Area function!

2. $\int_{-1}^{x} x+1\,dx = \left[\frac{1}{2}x^2 + x\right]_{-1}^{x} = \frac{1}{2}x^2 + x - \left(\frac{1}{2}(-1)^2 + -1\right) = \frac{1}{2}x^2 + x - \left(\frac{1}{2} - 1\right)$

$= \frac{1}{2}x^2 + x - \left(-\frac{1}{2}\right)$

$= \frac{1}{2}x^2 + x + \frac{1}{2}$ ← Area function!

3. $\int_{-1}^{x} 2x+3\,dx$

$= [x^2 + 3x]_{-1}^{x} = x^2 + 3x - ((-1)^2 + 3 \cdot -1)$

$x^2 + 3x - (1 - 3)$

$x^2 + 3x - (-2) \Rightarrow x^2 + 3x + 2$ ← Area function!

AP Calculus AB Name________________________

Definite Integrals Practice

Date________________

Evaluate each definite integral.

1) $\displaystyle\int_1^4 -\frac{1}{x^3}\,dx = \int_1^4 -x^{-3}\,dx$

$\left[\dfrac{\cancel{-}x^{-2}}{\cancel{-}2}\right]_1^4$

$\left[\dfrac{1}{2x^2}\right]_1^4 \Rightarrow \left[\dfrac{1}{2(4^2)} - \dfrac{1}{2\cdot 1^2}\right]$

$= \dfrac{1}{32} - \dfrac{1}{2} \Rightarrow \dfrac{-15}{32}$

2) $\displaystyle\int_1^4 (-x+2)\,dx = \left[-\frac{1}{2}x^2 + 2x\right]_1^4$

$-\frac{1}{2}(4)^2 + 2(4) - \left(-\frac{1}{2}\cdot 1^2 + 2(1)\right)$

$\cancel{-8+8} - \left(-\frac{1}{2} + 2\right)$

0

-1.5

3) $\displaystyle\int_0^3 (-2x-1)\,dx$

$\left[-x^2 - x\right]_0^3$

$-3^2 - 3 - \cancel{(-0^2 - 0)}$ 0

$-9 - 3 = -12$

4) $\displaystyle\int_{-3}^1 (-2x-2)\,dx = \left[-x^2 - 2x\right]_{-3}^1$

$-1^2 - 2(1) - \left(-(-3)^2 - 2(-3)\right)$

$-1 - 2 - (-9 + 6)$

$-3 - (-3) = -3 + 3 = 0$

5) $\displaystyle\int_1^4 -\frac{2}{x}\,dx$

6) $\displaystyle\int_1^3 (x^3 - 4x^2 + 4)\,dx$

7) $\displaystyle\int_{-3}^{-1} (2x^2 + 12x + 14)\,dx$

8) $\displaystyle\int_{-2}^{-1} \frac{2}{x^3}\,dx$

9) $\displaystyle\int_1^4 (x-1)\,dx$

10) $\displaystyle\int_{-4}^{-1} -\frac{4}{x}\,dx$