Honors Calculus: Study Guide – Beginning Applications Of Derivatives, Review

(Please take notes on separate sheet!)

Types of functions you need to be able to differentiate:

- Linear functions: the derivative is just the coefficient

1 ○ $Find \frac{dy}{dx} if\ y = -3(x-2)+9$

- Polynomial functions (Power Rule); involving radicals/exponents that can be re-written as polynomials

2 - $Find\ y' if\ y = \frac{\sqrt[3]{t^2}-3\sqrt{t}+2}{\sqrt{t}}$

- Products of functions: Use the product rule: $\frac{d}{dx}\big(f(x) * g(x)\big) =$ $f'(x) \cdot g(x) + f(x) \cdot g'(x)$
- Rational functions: Use the quotient rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) =$ $\frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$
- Composite functions: Use the chain rule: $\frac{d}{dx}(g(h(x)) =$ $g'(h(x)) \cdot h'(x)$
- Trigonometric functions:
  - ○ Ex: find the derivative of: $\sin(4x)$ $\cos(4x)$ $\tan(4x)$ $\csc(2x^2)$ $\sec(2x^2)$ $\cot(2x^2)$

$4\cos(4x)$ $-4\sin(4x)$ $4\sec^2(4x)$ $-4x\csc(2x^2)\cot(2x^2)$ ↓ $-4x\csc(2x^2)$

$4x\sec(2x)\tan(2x)$

3 - Exponential and logarithmic functions: What is the derivative of: $y = e^{\sin(x)}$ and $f(x) = 2x\ln(2x)$

a b

New Skills:

Related Rates

- Strategy: Translate all of the numbers given in terms of either rates ($\frac{d}{dt}$) or variables. Do the same for the rate you are looking for. Find/use a geometry formula that connects the rates and variables together. If necessary, eliminate an unnecessary variable by substitution. Then take the derivative with respect to time, plug in what you are given and solve for whatever unknown remains.

4 - A spherical snowball melts, so that its radius is decreasing by 4 cm per minute. How fast is the volume of the sphere changing when the radius is 6 cm? (V = $\frac{4}{3}\pi r^3$)

5 - A ladder 20 feet long leans against a building. If the bottom of the ladder slides away from the building horizontally at a rate of 4 ft/sec, how fast is the ladder sliding down the house when the top of the ladder is 8 feet from the ground?

6 - A water tank in the shape of a right circular cone has a height of 10 feet. The top rim of the tank is a circle with a radius of 4 feet. If water is being pumped into the tank at the rate of 2 cubic feet per minute, what is the rate of change of the water depth, in feet per minute, when the depth is 5 feet? (V = $\pi r^2 h$)

Critical Numbers

- Be able to take the derivative of a given function and determine its **critical numbers**.
  - ○ Strategy: Take the derivative and set it equal to zero; also consider any points where it is undefined (usually when the denominator, if any, is equal to zero).
  - ○ Ex: Find the critical numbers: a.) $f(x) = 6x^5 + 33x^4 - 30x^3 + 100$ and b.) $g(x) = \sqrt[3]{(x-2)^2} + 3$

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Relative Extrema

- Given a function, be able to find locations of relative maximums and minimums and justify your answer.
  - Strategy: Find critical numbers as before. Populate these values on a number line. Choose test values in every interval space created on the number line and determine polarity (sign) of number by plugging test value into the **derivative**. A critical number is merely a *candidate* for being a relative max or min: justification requires showing and explaining that the derivative changes from positive to negative for a max or negative to positive for a minimum. Use words to justify!

8 $y = \frac{16x}{x^2+16}$

9 $f(x) = x^3 - 2x^2 + 3$

10 $\cancel{x(t) = 2\cos\left(\frac{t}{2}\right) \; on \; [0,2\pi]}$ Not on test.

11 $y = -e^{2x}$

Absolute Extrema

- Given a function and a specified interval, determine absolute maximums and minimums.
  - Strategy: Find relative maxes and mins using strategy above. If you found any maxes or mins, plug those x-values into the **original function** to find the output y-values. Do the same thing for both the endpoints of the interval. Of the outputs, the biggest number is your absolute max, smallest = min.
  - Examples:

12 $f(x) = -x^3 + 4x^2 - 7 \; on \; [-1,2]$

13 $y = -\sqrt{-x+2} \; on \; [-4,1]$

14 $p(t) = 3t + \sin(4t) + 100 \; on \; [0,1]$

15 $y = 3x\sqrt[3]{(x+4)^2}$ on $[-5, 0]$

Review

- Write the equation of a line tangent to a curve at a point: ex: $f(x) = 2x^2 + 3x + 2$ at x = 1.
  - Strategy: You will need to use $y - y_1 = m(x - x_1)$. You are provided with $x_1 = 1$. Plug this into the function as provided to find y_1. Now all you need is m. Find $f'(x)$ and then plug the given x-value into it. Then place (x_1, y_1) and m into the appropriate places and you're done.

16

(1) $\frac{dy}{dx} = -3$ Derivative of linear fct. is just the slope / coefficient.

(2) $y = (t^{2/3} - 3t^{1/2} + 2)t^{-1/2}$

$y = y^{1/6} - 3 + 2y^{-1/2}$ Now take deriv.

$y' = \frac{1}{6}y^{-5/6} - \cancel{0} + \cancel{2} \cdot \frac{-1}{\cancel{2}} y^{-3/2}$

$y' = \frac{1}{6y^{5/6}} - \frac{1}{y^{3/2}}$

(3) a.) $y = e^{\sin(x)}$ chain rule

$\frac{dy}{dx} = e^{\sin(x)} \cdot \cos(x)$

b.) $y = \underbrace{\boxed{2x}}_{f} \cdot \underbrace{\boxed{\ln(2x)}}_{g}$ PRODUCT RULE $f' = 2$ $g' = \frac{1}{2x} \cdot 2 = \frac{\cancel{2}}{\cancel{2}x} = \frac{1}{x}$

$y'\ 2\ln(2x) + 2x \cdot \frac{1}{x}$

$y' = 2\ln(2x) + \frac{2\cancel{x}}{\cancel{x}}$

$\boxed{y' = 2\ln(2x) + 2}$

④ "Radius dec. by 4 cm/min" $\Rightarrow$ $\frac{dr}{dt} = -4$

"How fast is Volume changing?" $\Rightarrow$ $\frac{dV}{dt} = ?$ @ $r = 6cm$

GOAL

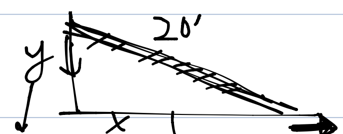
$$\frac{d}{dt}\, V = \left(\frac{4}{3}\pi r^3\right) \frac{d}{dt}$$

$$\frac{dV}{dt} = \frac{4\pi}{3} \cdot \cancel{3}r^2 \cdot \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi \cdot 6^2 \cdot -4$$

$$\frac{dV}{dt} = -576\pi \text{ cm}^3/\text{min}$$

⑤ 

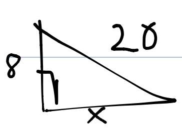
?

$\frac{dx}{dt} = 4$ GOAL $\frac{dy}{dt} = ?$ @ $y=8$

$$\frac{d}{dt}\left(x^2 + y^2\right) = \left(\cancel{20^2}\ 400\right)\frac{d}{dt}$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

? 4 8 ?



$8^2 + x^2 = 20^2$

$64 + x^2 = 400$

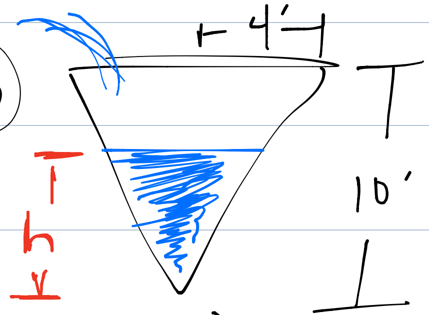
$x^2 = 336$

$x = \sqrt{336} \approx 18.330$

$$2(18.330)(4) + 2(8)\frac{dy}{dt} = 0$$

$$\frac{\cancel{16}\frac{dy}{dt}}{\cancel{16}} = \frac{-146.642}{16} \Rightarrow \boxed{\frac{dy}{dt} = -9.165 \text{ ft/sec}}$$

(8)

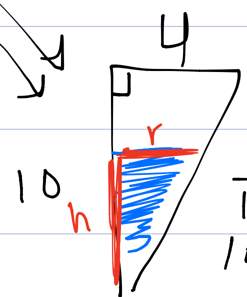


"Water is being pumped in..."

$$\Rightarrow \frac{dV}{dt} = 2 \text{ ft}^3/\text{min}$$

"Rate of change of water depth..."

$$\Rightarrow \underbrace{\frac{dh}{dt}}_{\text{GOAL}} = ? \quad @\, h = 5$$



$$\frac{4}{10} = \frac{r}{h}$$

$$10r = 4h$$

$$r = \frac{4h}{10}$$

$$r = \frac{2h}{5}$$

$$r^2 = \frac{4h^2}{25}$$

$$V = \frac{1}{3}\pi r^2 \cdot h$$

$$V = \frac{\pi}{3} \frac{4h^2}{25} \cdot h$$

$$\frac{d}{dt}\left(V = \frac{4\pi}{75} h^3\right)$$

$$\frac{dV}{dt} = \frac{4\pi}{\cancel{75}_{25}} \cdot \cancel{3}h^2 \cdot \frac{dh}{dt}$$

$$2 = \frac{4\pi}{25}(5)^2 \cdot \frac{dh}{dt}$$

$$2 = \frac{4\pi}{\cancel{25}} \cdot \cancel{25} \cdot \frac{dh}{dt}$$

$$\frac{2}{4\pi} = \frac{\cancel{4\pi}\frac{dh}{dt}}{\cancel{4\pi}}$$

$$\boxed{\frac{1}{2\pi} = \frac{dh}{dt} \quad \text{ft/min}}$$

7.) Critical #'s

$f(x) = 6x^5 + 33x^4 - 30x^3 + 100$

$f'(x) = 30x^4 + 132x^3 - 90x^2 = 0$

$6x^2(5x^2 + 22x - 15) = 0$

$6x^2(5x - 3)(x + 5) = 0$

$6x^2 = 0 \qquad 5x - 3 = 0 \qquad x + 5 = 0$

$\underline{x = 0} \qquad \underline{x = 3/5} \qquad \underline{x = -5}$ C.N.

b.) $g(x) = \sqrt[3]{(x-2)^2} + 3$

$g(x) = (x-2)^{2/3} + 3$

$g'(x) = \frac{2}{3}(x-2)^{-1/3} \cdot 1 + 0$

$g'(x) = \dfrac{2}{3\sqrt[3]{x-2}}$

$3\sqrt[3]{x-2} = 0 \qquad 2 = 0$

$\cancel{\emptyset}$

$\underline{\underline{x = 2}}$ C.N.

8.

$$\frac{16x}{x^2+16} \qquad \begin{array}{l} f' = 16 \\ g' = 2x \end{array}$$

$$\frac{16(x^2+16) - 16x(2x)}{(x^2+16)^2}$$

$$\frac{16x^2 + 256 - 32x^2}{(x^2+16)^2}$$

$$\frac{-16x^2+256}{(x^2+16)^2} = 0$$

$(x^2+16)^2 \rightarrow$ never 0

$$-16x^2 + 256 = 0$$

$$\frac{256}{16} = \frac{16x^2}{16}$$

$$16 = x^2$$

$$\underline{\underline{x = \pm 4}}$$

	$x < -4$	-4	$-4 < x < 4$	4	$x > 4$
f'	$-$		$+$		$-$

Rel min @ $x=-4$; Rel max @ $x=4$

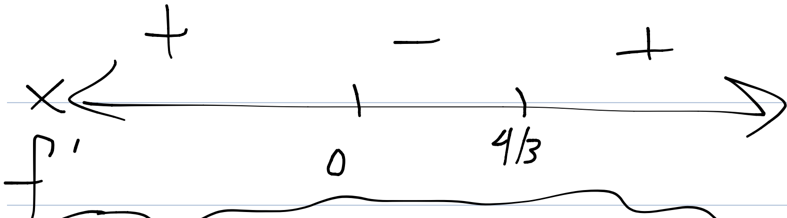
(9) $f(x) = x^3 - 2x^2 + 3$

$f'(x) = 3x^2 - 4x = 0$

$x(3x - 4) = 0$

$x = 0$ $\qquad$ $3x - 4 = 0$

$\qquad\qquad x = 4/3$



test-values into $f'(x)$

Rel. max @ $x = 0$ b/c f changes from inc. to decreasing here

Rel. min @ $x = 4/3$ b/c f changes from decreasing to increasing here.

11.) $y = -e^{2x}$

$\frac{dy}{dx} = -e^{2x} \cdot 2 = -2e^{2x} = 0$

$e^{2x} = 0$ no solution

y has no relative extrema b/c. no C.N. exists.

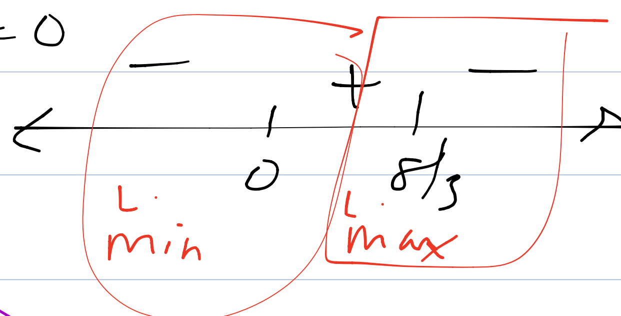
12. Absolute Ex.

$f(x) = -x^3 + 4x^2 - 7$ on $[-1, 2]$

$f'(x) = -3x^2 + 8x = 0$

$-x(3x - 8) = 0$

$x = 0$ $\quad$ $x = 8/3$



C.N.

$f(0) = -7$

$f(8/3) = 67/27 \approx 2.481$

E.P.

$f(-1) = -2$

$f(2) = 1$

Abs. max @ $x = 8/3$

Abs min. @ $x = 0$

(13) $y = -\sqrt{-x+2}$

$y = -(-x+2)^{1/2}$

$y' = -\frac{1}{2}(-x+2)^{-1/2} \cdot -1$

$y' = \frac{1}{2}(-x+2)^{-1/2}$

$y' = \dfrac{1}{2\sqrt{-x+2}} = 0$

$2\sqrt{x+2} = 0$

$-x+2=0$

$\underline{x = 2}$ C.N.

f' $+$ $\varnothing$

x 0

No Rel. ext.

<u>E.P.</u>

$f(-4) = -2.449$

$f(1) = -1$

<u>Abs max</u>

@ $x = 1$

<u>Abs min</u>

@ $x = -4$

14 $p(t) = 3t + \sin(4t) + 100 \quad [0, 1]$

$p'(t) = 3 + 4\cos(4t) = 0$

$4\cos(4t) = -3$

$\cos(4t) = -3/4$

$4t = \arccos(-3/4)$

$\frac{\cancel{4}t}{\cancel{4}} = \frac{2.418}{4}$

$t = 0.605$

$p'(t)$ $+$ $-$

t $\longleftarrow$ 0.605 $\longrightarrow$

max

$p(0.605) = 2.476$

$p(0) = 0$

$p(1) = 2.243$

$\boxed{\begin{array}{l} \text{Abs. max @ } x = 0.605 \\ \text{Abs min @ } x = 0 \end{array}}$

15. $y = 3x\sqrt[3]{(x+4)^2}$

$y = \underbrace{\boxed{3x}}_{f}\,\underbrace{\boxed{(x+4)^{2/3}}}_{g}$

$f' = 3$

$g' = \frac{2}{3}(x+4)^{-1/3} \cdot 1$

$g' = \frac{2}{3}(x+4)^{-1/3}$

↓ Prod. Rule

$y' = 3 \cdot (x+4)^{2/3} + \cancel{3}x \cdot \frac{2}{\cancel{3}}(x+4)^{-1/3}$

$3(x+4)^{2/3} + 2x(x+4)^{-1/3} = 0$

$(x+4)^{-1/3}\left[3(x+4) + 2x\right] = 0$

$\frac{1}{\sqrt[3]{x+4}}\left[3x + 12 + 2x\right] = 0$

$\frac{5x+12}{\sqrt[3]{x+4}} = 0$

$5x + 12 = 0$

$x = -12/5 = -2.4$

$\sqrt[3]{x+4} = 0$

$x = -4$

y'	+		−		+
x		-4		-2.4	
		l. max		l. min	

$$\left.\begin{aligned} f(-4) &= 0 \\ f(-2.4) &= -9.849 \\ f(-5) &= -15 \\ f(0) &= 0 \end{aligned}\right\}$$

$$\boxed{\begin{array}{l} \underline{\text{Abs. max}} \\ @\ x = -4,\ x = 0 \\ \underline{\text{Abs min}} \\ @\ x = -5 \end{array}}$$

16. $f(x) = 2x^2 + 3x + 2 \quad @ \quad x = 1$

<u>Point</u>:

$f(1) = 2 + 3 + 2 = 7 \Rightarrow \boxed{(1, 7)}$

<u>Slope</u>

$f'(x) = 4x + 3$

$f'(1) = 4 + 3 = \boxed{7 = m}$

$y - 7 = 7(x - 1)$

$y - \cancel{7} = 7x - \cancel{7}$

$\boxed{y = 7x}$