4.3 Connecting f' and f'' with the Graph of f

What you'll learn about

- First Derivative Test for Local Extrema
- Concavity
- Points of Inflection
- Second Derivative Test for Local Extrema
- Learning about Functions from Derivatives

. . . and why

Differential calculus is a powerful problem-solving tool precisely because of its usefulness for analyzing functions.

First Derivative Test for Local Extrema

As we see once again in Figure 4.18, a function f may have local extrema at some critical points while failing to have local extrema at others. The key is the sign of f' in a critical point's immediate vicinity. As x moves from left to right, the values of f increase where $f' > 0$ and decrease where $f' < 0$.

At the points where f has a minimum value, we see that $f' < 0$ on the interval immediately to the left and $f' > 0$ on the interval immediately to the right. (If the point is an endpoint, there is only the interval on the appropriate side to consider.) This means that the curve is falling (values decreasing) on the left of the minimum value and rising (values increasing) on its right. Similarly, at the points where f has a maximum value, $f' > 0$ on the interval immediately to the left and $f' < 0$ on the interval immediately to the right. This means that the curve is rising (values increasing) on the left of the maximum value and falling (values decreasing) on its right.

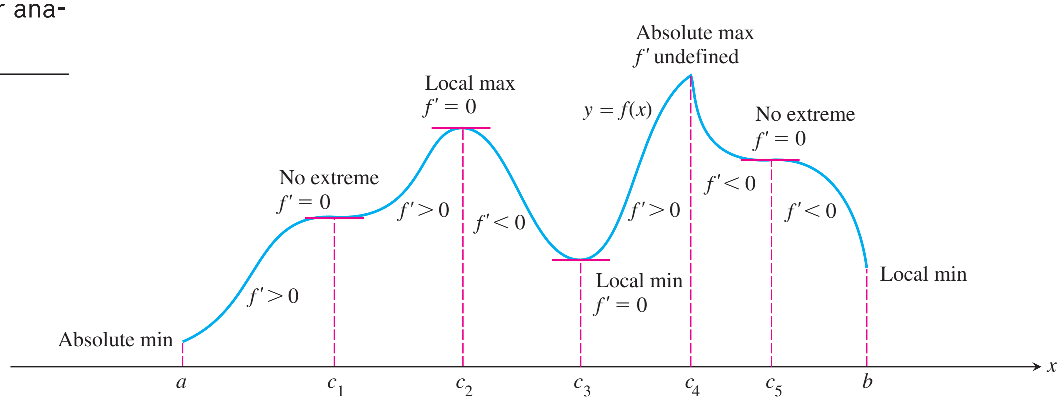


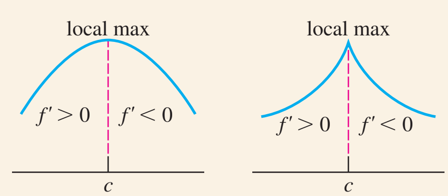
Figure 4.18 A function's first derivative tells how the graph rises and falls.

THEOREM 4 First Derivative Test for Local Extrema

The following test applies to a continuous function $f(x)$.

At a critical point c:

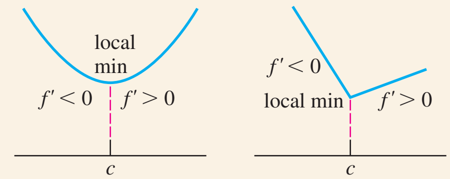
1. If f' changes sign from positive to negative at c ($f' > 0$ for $x < c$ and $f' < 0$ for $x > c$), then f has a local maximum value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

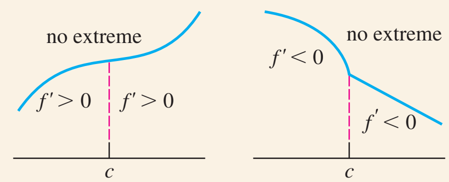
continued

2. If f' changes sign from negative to positive at c ($f' < 0$ for $x < c$ and $f' > 0$ for $x > c$), then f has a local minimum value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

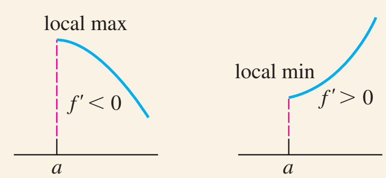
3. If f' does not change sign at c (f' has the same sign on both sides of c), then f has no local extreme value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

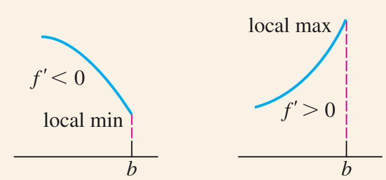
At a left endpoint *a*:

If $f' < 0$ ($f' > 0$) for $x > a$, then f has a local maximum (minimum) value at a.



At a right endpoint *b*:

If $f' < 0$ ($f' > 0$) for $x < b$, then f has a local minimum (maximum) value at b.



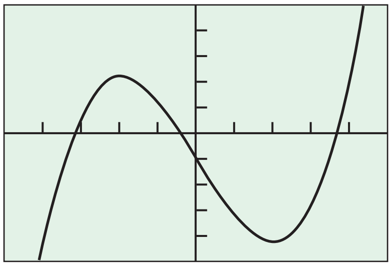
Here is how we apply the First Derivative Test to find the local extrema of a function. The critical points of a function f partition the x-axis into intervals on which f' is either positive or negative. We determine the sign of f' in each interval by evaluating f' for one value of x in the interval. Then we apply Theorem 4 as shown in Examples 1 and 2.

EXAMPLE 1 Using the First Derivative Test

For each of the following functions, use the First Derivative Test to find the local extreme values. Identify any absolute extrema.

(a) $f(x) = x^3 - 12x - 5$ **(b)** $g(x) = (x^2 - 3)e^x$

continued

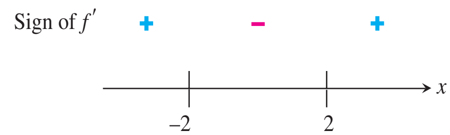


[–5, 5] by [–25, 25]

Figure 4.19 The graph of $f(x) = x^3 - 12x - 5$.

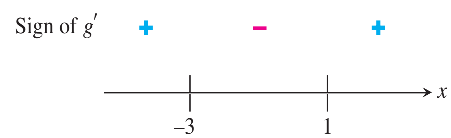
SOLUTION

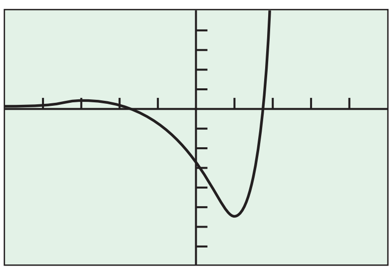
(a) Since f is differentiable for all real numbers, the only possible critical points are the zeros of f'. Solving $f'(x) = 3x^2 - 12 = 0$, we find the zeros to be $x = 2$ and $x = -2$. The zeros partition the x-axis into three intervals, as shown below:



Using the First Derivative Test, we can see from the sign of f' on each interval that there is a local maximum at $x = -2$ and a local minimum at $x = 2$. The local maximum value is $f(-2) = 11$, and the local minimum value is $f(2) = -21$. There are no absolute extrema, as the function has range $(-\infty, \infty)$ (Figure 4.19).

(b) Since g is differentiable for all real numbers, the only possible critical points are the zeros of g'. Since $g'(x) = (x^2 - 3) \cdot e^x + (2x) \cdot e^x = (x^2 + 2x - 3) \cdot e^x$, we find the zeros of g' to be $x = 1$ and $x = -3$. The zeros partition the x-axis into three intervals, as shown below:





[–5, 5] by [–8, 5]

Figure 4.20 The graph of $g(x) = (x^2 - 3)e^x$.

Using the First Derivative Test, we can see from the sign of f' on each interval that there is a local maximum at $x = -3$ and a local minimum at $x = 1$. The local maximum value is $g(-3) = 6e^{-3} \approx 0.299$, and the local minimum value is $g(1) = -2e \approx -5.437$. Although this function has the same increasing–decreasing–increasing pattern as f, its left end behavior is quite different. We see that $\lim_{x\to -\infty} g(x) = 0$, so the graph approaches the y-axis asymptotically and is therefore bounded below. This makes $g(1)$ an *absolute* minimum. Since $\lim_{x\to\infty} g(x) = \infty$, there is no absolute maximum (Figure 4.20).

Now try Exercise 3.

Concavity

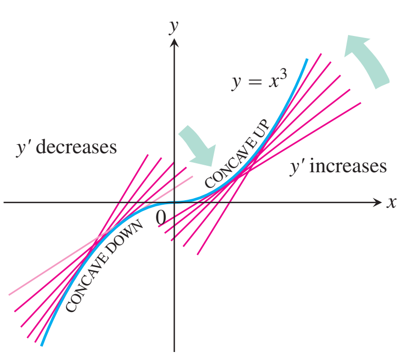


Figure 4.21 The graph of $y = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

As you can see in Figure 4.21, the function $y = x^3$ rises as x increases, but the portions defined on the intervals $(-\infty, 0)$ and $(0, \infty)$ *turn* in different ways. Looking at tangents as we scan from left to right, we see that the slope y' of the curve decreases on the interval $(-\infty, 0)$ and then increases on the interval $(0, \infty)$. The curve $y = x^3$ is *concave down* on $(-\infty, 0)$ and *concave up* on $(0, \infty)$. The curve lies below the tangents where it is concave down, and above the tangents where it is concave up.

DEFINITION Concavity

The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if y' is increasing on I.

(b) **concave down** on an open interval I if y' is decreasing on I.

If a function $y = f(x)$ has a second derivative, then we can conclude that y' increases if $y'' > 0$ and y' decreases if $y'' < 0$.

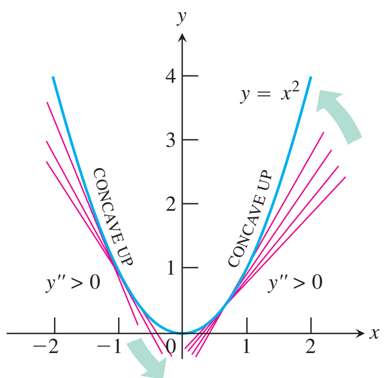
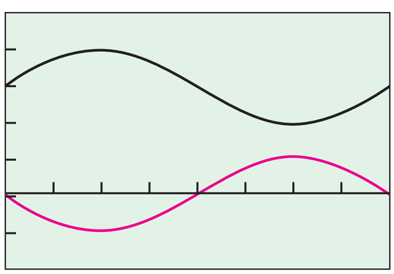


Figure 4.22 The graph of $y = x^2$ is concave up on any interval. (Example 2)

$y_1 = 3 + \sin x$, $y_2 = -\sin x$



[0, 2π] by [–2, 5]

Figure 4.23 Using the graph of y'' to determine the concavity of y. (Example 2)

> **Concavity Test**
>
> The graph of a twice-differentiable function $y = f(x)$ is
>
> **(a)** concave up on any interval where $y'' > 0$.
>
> **(b)** concave down on any interval where $y'' < 0$.

EXAMPLE 2 Determining Concavity

Use the Concavity Test to determine the concavity of the given functions on the given intervals:

(a) $y = x^2$ on $(3, 10)$ **(b)** $y = 3 + \sin x$ on $(0, 2\pi)$

SOLUTION

(a) Since $y'' = 2$ is always positive, the graph of $y = x^2$ is concave up on *any* interval. In particular, it is concave up on $(3, 10)$ (Figure 4.22).

(b) The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is positive (Figure 4.23).

Now try Exercise 7.

Points of Inflection

The curve $y = 3 + \sin x$ in Example 2 changes concavity at the point $(\pi, 3)$. We call $(\pi, 3)$ a *point of inflection* of the curve.

> **DEFINITION Point of Inflection**
>
> A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection.**

A point on a curve where y'' is positive on one side and negative on the other is a point of inflection. At such a point, y'' is either zero (because derivatives have the intermediate value property) or undefined. If y is a twice differentiable function, $y'' = 0$ at a point of inflection and y' has a local maximum or minimum.

EXAMPLE 3 Finding Points of Inflection

Find all points of inflection of the graph of $y = e^{-x^2}$.

SOLUTION

First we find the second derivative, recalling the Chain and Product Rules:

$$y = e^{-x^2}$$

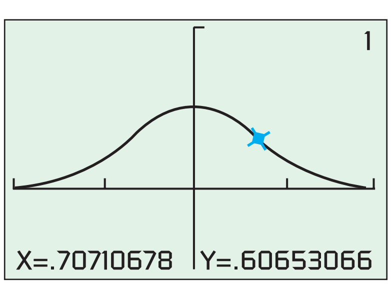
$$y' = e^{-x^2} \cdot (-2x)$$

$$y'' = e^{-x^2} \cdot (-2x) \cdot (-2x) + e^{-x^2} \cdot (-2)$$

$$= e^{-x^2}(4x^2 - 2)$$

The factor e^{-x^2} is always positive, while the factor $(4x^2 - 2)$ changes sign at $-\sqrt{1/2}$ and at $\sqrt{1/2}$. Since y'' must also change sign at these two numbers, the points of inflection are $(-\sqrt{1/2}, 1/\sqrt{e})$ and $(\sqrt{1/2}, 1/\sqrt{e})$. We confirm our solution graphically by observing the changes of curvature in Figure 4.24.

Now try Exercise 13.



[–2, 2] by [–1, 2]

Figure 4.24 Graphical confirmation that the graph of $y = e^{-x^2}$ has a point of inflection at $x = \sqrt{1/2}$ (and hence also at $x = -\sqrt{1/2}$). (Example 3)

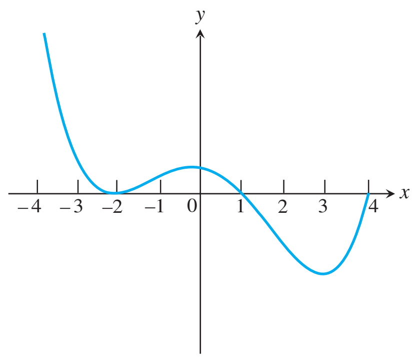


Figure 4.25 The graph of f', the derivative of f, on the interval $[-4, 4]$.

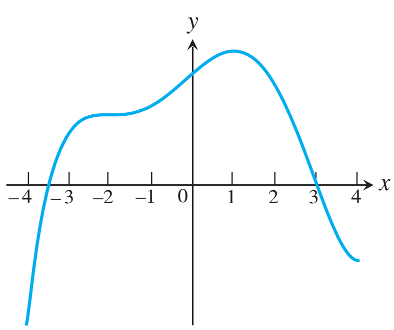
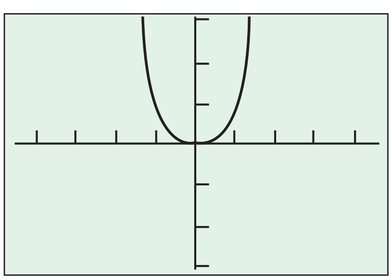
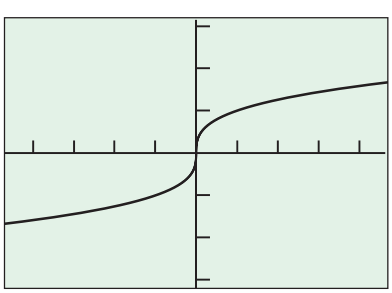


Figure 4.26 A possible graph of f. (Example 4)



[– 4.7, 4.7] by [–3.1, 3.1]

Figure 4.27 The function $f(x) = x^4$ does not have a point of inflection at the origin, even though $f''(0) = 0$.



[–4.7, 4.7] by [–3.1, 3.1]

Figure 4.28 The function $f(x) = \sqrt[3]{x}$ has a point of inflection at the origin, even though $f''(0) \neq 0$.

EXAMPLE 4 Reading the Graph of the Derivative

The graph of the *derivative* of a function f on the interval $[-4, 4]$ is shown in Figure 4.25. Answer the following questions about f, justifying each answer with information obtained from the graph of f'.

(a) On what intervals is f increasing?

(b) On what intervals is the graph of f concave up?

(c) At which x-coordinates does f have local extrema?

(d) What are the x-coordinates of all inflection points of the graph of f?

(e) Sketch a possible graph of f on the interval $[-4, 4]$.

SOLUTION

(a) Since $f' > 0$ on the intervals $[-4, -2)$ and $(-2, 1)$, the function f must be increasing on the entire interval $[-4, 1]$ with a horizontal tangent at $x = -2$ (a "shelf point").

(b) The graph of f is concave up on the intervals where f' is increasing. We see from the graph that f' is increasing on the intervals $(-2, 0)$ and $(3, 4)$.

(c) By the First Derivative Test, there is a local maximum at $x = 1$ because the sign of f' changes from positive to negative there. Note that there is no extremum at $x = -2$, since f' does not change sign. Because the function increases from the left endpoint and decreases to the right endpoint, there are local minima at the endpoints $x = -4$ and $x = 4$.

(d) The inflection points of the graph of f have the same x-coordinates as the turning points of the graph of f', namely -2, 0, and 3.

(e) A possible graph satisfying all the conditions is shown in Figure 4.26.

Now try Exercise 23.

Caution: It is tempting to oversimplify a point of inflection as a point where the second derivative is zero, but that can be wrong for two reasons:

1. *The second derivative can be zero at a noninflection point*. For example, consider the function $f(x) = x^4$ (Figure 4.27). Since $f''(x) = 12x^2$, we have $f''(0) = 0$; however, $(0, 0)$ is not an inflection point. Note that f'' does not *change sign* at $x = 0$.
2. *The second derivative need not be zero at an inflection point*. For example, consider the function $f(x) = \sqrt[3]{x}$ (Figure 4.28). The concavity changes at $x = 0$, but there is a *vertical* tangent line, so both $f'(0)$ and $f''(0)$ fail to exist.

Therefore, the only safe way to test algebraically for a point of inflection is to confirm a sign change of the second derivative. This *could* occur at a point where the second derivative is zero, but it also could occur at a point where the second derivative fails to exist.

To study the motion of a body moving along a line, we often graph the body's position as a function of time. One reason for doing so is to reveal where the body's acceleration, given by the second derivative, changes sign. On the graph, these are the points of inflection.

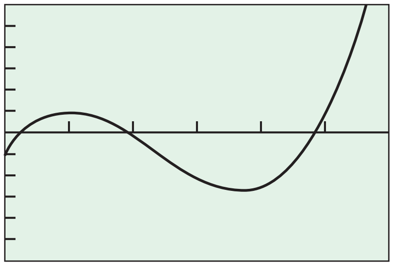
EXAMPLE 5 Studying Motion along a Line

A particle is moving along the x-axis with position function

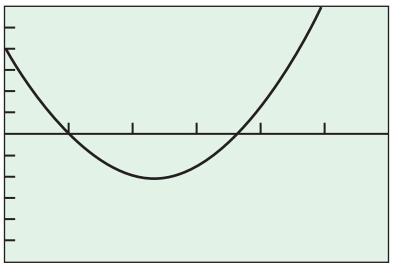
$$x(t) = 2t^3 - 14t^2 + 22t - 5, \qquad t \geq 0.$$

Find the velocity and acceleration, and describe the motion of the particle.

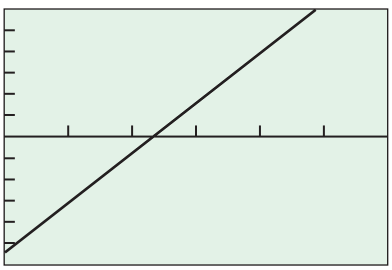
continued



[0, 6] by [–30, 30]
(a)



[0, 6] by [–30, 30]
(b)



[0, 6] by [–30, 30]
(c)

Figure 4.29 The graph of
(a) $x(t) = 2t^3 - 14t^2 + 22t - 5$, $t \geq 0$,
(b) $x'(t) = 6t^2 - 28t + 22$, and
(c) $x''(t) = 12t - 28$. (Example 5)

SOLUTION

Solve Analytically

The velocity is

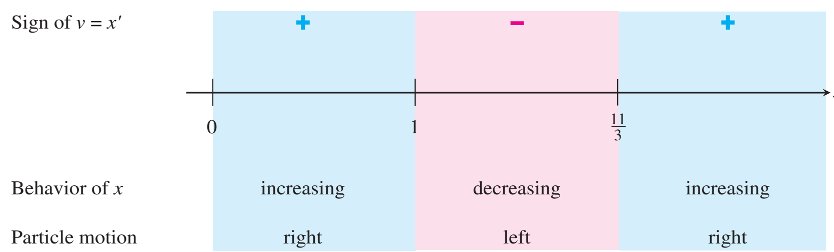
$$v(t) = x'(t) = 6t^2 - 28t + 22 = 2(t - 1)(3t - 11),$$

and the acceleration is

$$a(t) = v'(t) = x''(t) = 12t - 28 = 4(3t - 7).$$

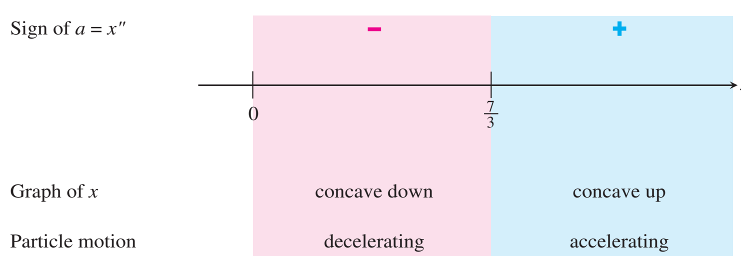
When the function $x(t)$ is increasing, the particle is moving to the right on the x-axis; when $x(t)$ is decreasing, the particle is moving to the left. Figure 4.29 shows the graphs of the position, velocity, and acceleration of the particle.

Notice that the first derivative ($v = x'$) is zero when $t = 1$ and $t = 11/3$. These zeros partition the t-axis into three intervals, as shown in the sign graph of v below:



The particle is moving to the right in the time intervals [0, 1) and (11/3, ∞) and moving to the left in (1, 11/3).

The acceleration $a(t) = 12t - 28$ has a single zero at $t = 7/3$. The sign graph of the acceleration is shown below:



The accelerating force is directed toward the left during the time interval [0, 7/3], is momentarily zero at $t = 7/3$, and is directed toward the right thereafter.

Now try Exercise 25.

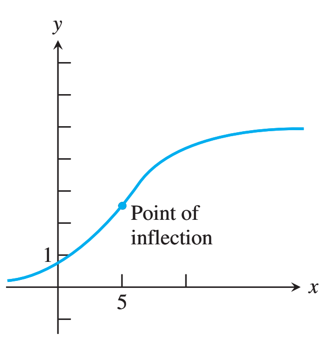


Figure 4.30 A logistic curve

$$y = \frac{c}{1 + ae^{-bx}}.$$

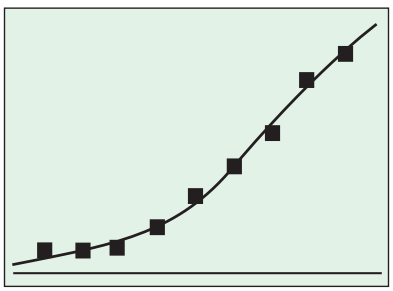
The growth of an individual company, of a population, in sales of a new product, or of salaries often follows a *logistic* or *life cycle curve* like the one shown in Figure 4.30. For example, sales of a new product will generally grow slowly at first, then experience a period of rapid growth. Eventually, sales growth slows down again. The function f in Figure 4.30 is increasing. Its rate of increase, f', is at first increasing ($f'' > 0$) up to the point of inflection, and then its rate of increase, f', is decreasing ($f'' < 0$). This is, in a sense, the opposite of what happens in Figure 4.21.

Some graphers have the logistic curve as a built-in regression model. We use this feature in Example 6.

Table 4.2 Population of Alaska

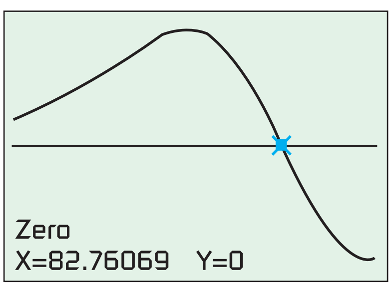
Years since 1900	Population
20	55,036
30	59,278
40	75,524
50	128,643
60	226,167
70	302,583
80	401,851
90	550,043
100	626,932

Source: Bureau of the Census, U.S. Chamber of Commerce.



[12, 108] by [0, 730000]

(a)



[12, 108] by [–250, 250]

(b)

Figure 4.31 (a) The logistic regression curve

$$y = \frac{895598}{1 + 71.57e^{-0.0516x}}$$

superimposed on the population data from Table 4.2, and (b) the graph of y'' showing a zero at about $x = 83$.

EXAMPLE 6 Population Growth in Alaska

Table 4.2 shows the population of Alaska in each 10-year census between 1920 and 2000.

(a) Find the logistic regression for the data.

(b) Use the regression equation to predict the Alaskan population in the 2020 census.

(c) Find the inflection point of the regression equation. What significance does the inflection point have in terms of population growth in Alaska?

(d) What does the regression equation indicate about the population of Alaska in the long run?

SOLUTION

(a) Using years since 1900 as the independent variable and population as the dependent variable, the logistic regression equation is approximately

$$y = \frac{895598}{1 + 71.57e^{-0.0516x}}.$$

Its graph is superimposed on a scatter plot of the data in Figure 4.31(a). Store the regression equation as Y1 in your calculator.

(b) The calculator reports Y1(120) to be approximately 781,253. (Given the uncertainty of this kind of extrapolation, it is probably more reasonable to say "approximately 781,200.")

(c) The inflection point will occur where y'' changes sign. Finding y'' algebraically would be tedious, but we can graph the numerical derivative of the numerical derivative and find the zero graphically. Figure 4.31(b) shows the graph of y'', which is nDeriv(nDeriv(Y1,X,X),X,X) in calculator syntax. The zero is approximately 83, so the inflection point occurred in 1983, when the population was about 450,570 and growing the fastest.

(d) Notice that $\lim_{x\to\infty} \frac{895598}{1+ 71.57e^{-0.0516x}} = 895598$, so the regression equation indicates that the population of Alaska will stabilize at about 895,600 in the long run. Do not put too much faith in this number, however, as human population is dependent on too many variables that can, and will, change over time. ***Now try Exercise 31.***

Second Derivative Test for Local Extrema

Instead of looking for sign changes in y' at critical points, we can sometimes use the following test to determine the presence of local extrema.

THEOREM 5 Second Derivative Test for Local Extrema

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.

2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

This test requires us to know f'' *only at c itself* and not in an interval about c. This makes the test easy to apply. That's the good news. The bad news is that the test fails if $f''(c) = 0$ or if $f''(c)$ fails to exist. When this happens, go back to the First Derivative Test for local extreme values.

In Example 7, we apply the Second Derivative Test to the function in Example 1.

EXAMPLE 7 Using the Second Derivative Test

Find the local extreme values of $f(x) = x^3 - 12x - 5$.

SOLUTION

We have

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$
$$f''(x) = 6x.$$

Testing the critical points $x = \pm 2$ (there are no endpoints), we find

$$f''(-2) = -12 < 0 \Rightarrow f \text{ has a local maximum at } x = -2 \text{ and}$$
$$f''(2) = 12 > 0 \Rightarrow f \text{ has a local minimum at } x = 2.$$

Now try Exercise 35.

EXAMPLE 8 Using f' and f'' to Graph f

Let $f'(x) = 4x^3 - 12x^2$.

(a) Identify where the extrema of f occur.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

(c) Find where the graph of f is concave up and where it is concave down.

(d) Sketch a possible graph for f.

SOLUTION

f is continuous since f' exists. The domain of f' is $(-\infty, \infty)$, so the domain of f is also $(-\infty, \infty)$. Thus, the critical points of f occur only at the zeros of f'. Since

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3),$$

the first derivative is zero at $x = 0$ and $x = 3$.

Intervals	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

(a) Using the First Derivative Test and the table above we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.

(b) Using the table above we see that f is decreasing in $(-\infty, 0]$ and $[0, 3]$, and increasing in $[3, \infty)$.

(c) $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$.

Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f''	$+$	$-$	$+$
Behavior of f	concave up	concave down	concave up

We see that f is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$.

continued

Note

The *Second Derivative Test* does not apply at $x = 0$ because $f''(0) = 0$. We need the First Derivative Test to see that there is no local extremum at $x = 0$.

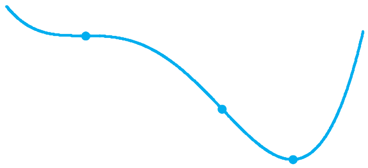


Figure 4.32 The graph for f has no extremum but has points of inflection where $x = 0$ and $x = 2$, and a local minimum where $x = 3$. (Example 8)

(d) Summarizing the information in the two tables above we obtain

$x < 0$	$0 < x < 2$	$2 < x < 3$	$x < 3$
decreasing	decreasing	decreasing	increasing
concave up	concave down	concave up	concave up

Figure 4.32 shows one possibility for the graph of f. ***Now try Exercise 39.***

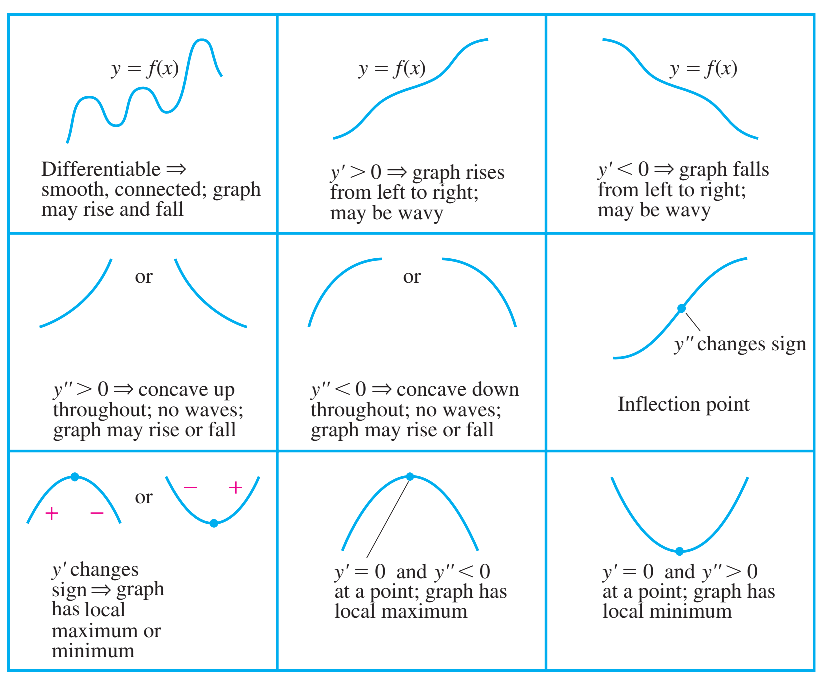
EXPLORATION 1 **Finding *f* from *f′***

Let $f'(x) = 4x^3 - 12x^2$.

1. Find three different functions with derivative equal to $f'(x)$. How are the graphs of the three functions related?
2. Compare their behavior with the behavior found in Example 8.

Learning about Functions from Derivatives

We have seen in Example 8 and Exploration 1 that we are able to recover almost everything we need to know about a differentiable function $y = f(x)$ by examining y'. We can find where the graph rises and falls and where any local extrema are assumed. We can differentiate y' to learn how the graph bends as it passes over the intervals of rise and fall. We can determine the shape of the function's graph. The only information we cannot get from the derivative is how to place the graph in the xy-plane. As we discovered in Section 4.2, the only additional information we need to position the graph is the value of f at one point.



Remember also that a function can be continuous and still have points of nondifferentiability (cusps, corners, and points with vertical tangent lines). Thus, a noncontinuous graph of f' could lead to a continuous graph of f, as Example 9 shows.

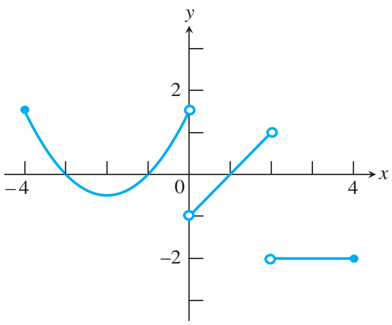


Figure 4.33 The graph of f', a discontinuous derivative.

EXAMPLE 9 Analyzing a Discontinuous Derivative

A function f is continuous on the interval $[-4, 4]$. The discontinuous function f', with domain $[-4, 0) \cup (0, 2) \cup (2, 4]$, is shown in the graph to the right (Figure 4.33).

(a) Find the x-coordinates of all local extrema and points of inflection of f.

(b) Sketch a possible graph of f.

SOLUTION

(a) For extrema, we look for places where f' changes sign. There are local maxima at $x = -3$, 0, and 2 (where f' goes from positive to negative) and local minima at $x = -1$ and 1 (where f' goes from negative to positive). There are also local minima at the two endpoints $x = -4$ and 4, because f' starts positive at the left endpoint and ends negative at the right endpoint.

For points of inflection, we look for places where f'' changes sign, that is, where the graph of f' changes direction. This occurs only at $x = -2$.

(b) A possible graph of f is shown in Figure 4.34. The derivative information determines the shape of the three components, and the continuity condition determines that the three components must be linked together. ***Now try Exercises 49 and 53.***

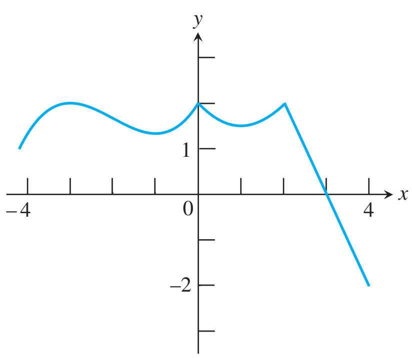


Figure 4.34 A possible graph of f. (Example 9)

EXPLORATION 2 Finding *f* from *f′* and *f″*

A function f is continuous on its domain $[-2, 4]$, $f(-2) = 5$, $f(4) = 1$, and f' and f'' have the following properties.

x	$-2 < x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$2 < x < 4$
f'	$+$	does not exist	$-$	0	$-$
f''	$+$	does not exist	$+$	0	$-$

1. Find where all absolute extrema of f occur.
2. Find where the points of inflection of f occur.
3. Sketch a possible graph of f.

Quick Review 4.3 *(For help, go to Sections 1.3, 2.2, 3.3, and 3.9.)*

In Exercises 1 and 2, factor the expression and use sign charts to solve the inequality.

1. $x^2 - 9 < 0$ $(-3, 3)$

2. $x^3 - 4x > 0$ $(-2, 0) \cup (2, \infty)$

In Exercises 3–6, find the domains of f and f'.

3. $f(x) = xe^x$ f: all reals; f': all reals

4. $f(x) = x^{3/5}$ f: all reals; f': $x \neq 0$

5. $f(x) = \dfrac{x}{x-2}$ f: $x \neq 2$; f': $x \neq 2$

6. $f(x) = x^{2/5}$ f: all reals; f': $x \neq 0$

In Exercises 7–10, find the horizontal asymptotes of the function's graph.

7. $y = (4 - x^2)e^x$ $y = 0$

8. $y = (x^2 - x)e^{-x}$ $y = 0$

9. $y = \dfrac{200}{1 + 10e^{-0.5x}}$ $y = 0$ and $y = 200$

10. $y = \dfrac{750}{2 + 5e^{-0.1x}}$ $y = 0$ and $y = 375$